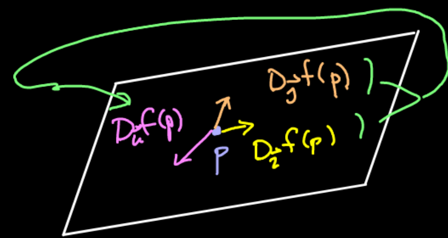
Linear Approximation: Tangent Planes

Idea: Give a (first-order) approximation of a function by a linear object (cf. tangent lines in Calculus I).

A function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is well-approximated by a plane near $p = (x_0, y_0) \in D$ if and only if the directional derivative at (x_0, y_0) in the direction of every $\vec{u} = \langle u_1, u_2 \rangle$ is a linear combination of the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, using the vector $\frac{\vec{u}}{|\vec{u}|}$ for scaling... Formally, this means



$$\to D_{\vec{u}} f(p) = \vec{\nabla} f(p) \cdot \frac{\vec{u}}{|\vec{u}|} \quad \text{for all nonzero } \vec{u}.$$

Point: A function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is *differentiable* at p when $D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$ for all unit vectors $\vec{u} \in \mathbb{R}^2$.

Even better, we can then give a formula for the tangent plane to the function f at (x_0, y_0), using the Implicit Function Theorem. Indeed, $f(x,y) = z$ yields $F(x,y,z) = f(x,y) - z$, with $F(x_0, y_0, f(x_0,y_0)) = 0$.

Thus
$$0 = \nabla F(x_0, y_0, f(x_0,y_0)) \cdot (\vec{x} - \langle x_0, y_0, f(x_0,y_0) \rangle)$$
$$0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - \underset{= z_0}{f(x_0,y_0)})$$

Is the equation of the desired tangent plane.

Ex: Compute an equation of the tangent plane to the given surface at the given point.

① $z = 2x^2 + y^2 - 5y$ at $(1, 1, -2) = (x_0, y_0, z_0)$

Sol: Tangent plane: $z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f_x = 4x \quad \therefore f_x(1,1) = 4 \qquad f_y = 2y - 5 \quad \therefore f_y(1,1) = -3$

Hence the desired tangent plane has formula

$$z - (-2) = 4(x-1) - 3(y-1)$$

☑

② $z = x\sin(x+y)$ at $\left(1, \frac{\pi - 3}{3}\right)$.

Sol: Tangent plane: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

$f_x = x\cos(x+y) + \sin(x+y) \quad \therefore f_x\left(1, \frac{\pi-3}{3}\right) = \frac{1+\sqrt{3}}{2}$, $\quad f_y = x\cos(x+y) \therefore f_y\left(1,\frac{\pi-3}{3}\right) = \frac{1}{2}$

$f(x_0,y_0) = \frac{\sqrt{3}}{2}$

Hence the tangent plane has formula:

$$z - \frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2}(x-1) + \frac{1}{2}\left(y - \frac{\pi-3}{3}\right)$$

☑

NB: Everything we've done so far will also make sense for functions of n variables. The key difference is that the result is a tangent space of dimension n (i.e. a vector space of dimension n) as opposed to a tangent plane; these sit in $\mathbb{R}^{n+1}$ as tangent hyperplanes to the graph of the resulting function...

We'll just stick to functions of two and three variables...

Ex: Compute the tangent plane to ...

① $z = e^{x-y}$ when $x = -1 = y$.

Sol: Tangent Plane: $z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f(x_0,y_0) = e^{(-1)-(-1)} = 1$, $\quad$ (NB: $f(x,y) = e^{x-y} = e^{x} \cdot e^{-y}$)

$f_x = e^{x-y}$ $\quad \therefore f_x(-1,-1) = e^{(-1)-(-1)} = 1$

$f_y = -e^{x-y}$ $\quad \therefore f_y(-1,-1) = -e^{(-1)-(-1)} = -1$

$\therefore$ The tangent plane is $z - 1 = 1(x-(-1)) - 1(y-(-1))$

i.e. $z - 1 = (x+1) - (y+1)$ ////

② $f(x,y,z) = z\ln(x^2 - y^2)$ at $(3,-1,2)$. $\quad f = w$

Sol: Tangent Hyperplane: $w - w_0 = \nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle$

$w_0 = f(x_0,y_0,z_0) = f(3,-1,2) = 2\ln(3^2-(-1)^2) = 2\ln(8) = 6\ln(2)$

$\nabla f = \left\langle \frac{2xz}{x^2-y^2}, -\frac{2yz}{x^2-y^2}, \ln(x^2-y^2) \right\rangle$

$\therefore \nabla f(x_0,y_0,z_0) = \nabla f(3,-1,2) = \left\langle \frac{2\cdot 3\cdot 2}{3^2-(-1)^2}, -\frac{2\cdot -1\cdot 2}{3^2-(-1)^2}, \ln(3^2-(-1)^2) \right\rangle = \left\langle \frac{3}{2}, \frac{1}{2}, 3\ln(2) \right\rangle$

$\therefore$ The tangent hyperplane has formula

$w - 6\ln(2) = \frac{3}{2}(x-3) + \frac{1}{2}(y+1) + 3\ln(2)(z-2)$ ////

③ $E^2 = p^2c^2 + m^2c^4$ at $(c,m,p,E) = (1,4,-3,5)$

Sol: normal vector to the tangent hyperplane is

$\nabla(p^2c^2 + m^2c^4 - E^2) = \langle 2cp^2 + 4m^2c^3, 2mc^4, 2pc^2, -2E \rangle$

$\therefore \vec{n} = 2\langle 1\cdot(-3)^2 + 2\cdot 4^2\cdot 1^3, 4\cdot 1^4, -3\cdot 1^2, -5 \rangle = 2\langle 41, 4, -3, -5 \rangle$

The desired tangent hyperplane is given by

$\langle 41,4,-3,-5 \rangle \cdot \langle c-1, m-4, p+3, E-5 \rangle = 0$; i.e.

$41(c-1) + 4(m-4) - 3(p+3) - 5(E-5) = 0$ ////

LINEAR APPROXIMATION

As an application of tangent planes, we can approximate a function locally by its tangent plane. In particular, we approximate

$$\Delta f \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n \qquad (\text{i.e. } \underset{\text{error}}{\Delta f} \approx \nabla f \cdot \Delta \vec{x})$$

NB: the approximation is better when $|\Delta \vec{x}|$ is small...

Def^n: The (total) differential of function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is

$$df := \nabla f \cdot d\vec{x} \quad \text{where} \quad d\vec{x} := \langle dx_1, dx_2, \cdots, dx_n \rangle.$$

(df: total differential; $dx_1, dx_2, \ldots, dx_n$: differential symbols)

Philosophy Above: $\Delta f \approx df$

Ex: Compute the differential of each function below.

① $f(x,y) = \sqrt{x^2+3y^2} = (x^2+3y^2)^{1/2}$

Sol: $\nabla f = \frac{1}{2}(x^2+3y^2)^{-1/2} \cdot \nabla(x^2+3y^2) = \frac{1}{2}(x^2+3y^2)^{-1/2}\langle 2x, 6y\rangle$

$$df = \nabla f \cdot d\vec{x} = (x^2+3y^2)^{-1/2}\langle x, 3y\rangle \cdot \langle dx, dy\rangle$$

$$= \frac{x}{\sqrt{x^2+3y^2}}\,dx + \frac{3y}{\sqrt{x^2+3y^2}}\,dy \qquad \boxed{//}$$

② $T(u,v,w) = \dfrac{v}{1+uvw} = v(1+uvw)^{-1}$

Sol: $\nabla T = \langle -v^2w(1+uvw)^{-2},\ v(-uw(1+uvw)^{-2}) + (1+uvw)^{-1},\ -uv^2(1+uvw)^{-2}\rangle$

$$= (1+uvw)^{-2}\langle -v^2w,\ 1+uvw-uvw,\ -uv^2\rangle = \frac{1}{(1+uvw)^2}\langle -v^2w, 1, -uv^2\rangle$$

$$\therefore dT = \nabla T \cdot d\vec{x} = \frac{-v^2w}{(1+uvw)^2}\,du + \frac{1}{(1+uvw)^2}\,dv - \frac{uv^2}{(1+uvw)^2}\,dw \qquad \boxed{//}$$

③ $R(\alpha, \beta, \gamma) = \alpha\beta^2\cos(\gamma)$

Sol: $\nabla R = \left\langle \frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta}, \frac{\partial R}{\partial \gamma} \right\rangle = \left\langle \beta^2\cos(\gamma), 2\alpha\beta\cos(\gamma), -\alpha\beta^2\sin(\gamma) \right\rangle$

$\therefore dR = \nabla R \cdot d\vec{\alpha} = \left\langle \beta^2\cos(\gamma), 2\alpha\beta\cos(\gamma), -\alpha\beta^2\sin(\gamma) \right\rangle \cdot \langle d\alpha, d\beta, d\gamma \rangle$

$= \beta^2\cos(\gamma)\,d\alpha + 2\alpha\beta\cos(\gamma)\,d\beta - \alpha\beta^2\sin(\gamma)\,d\gamma.$ //

④ $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$

Sol: $dx = \nabla x \cdot d\vec{r}$

$\nabla x = \left\langle \frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta} \right\rangle$

$= \langle \cos(\theta), -r\sin(\theta) \rangle$

$\therefore dx = \cos(\theta)\,dr - r\sin(\theta)\,d\theta$

$dy = \nabla y \cdot d\vec{r}$

$\nabla y = \left\langle \frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta} \right\rangle$

$= \langle \sin(\theta), r\cos(\theta) \rangle$

$\therefore dy = \sin(\theta)\,dr + r\cos(\theta)\,d\theta.$ //

Ex: Approximate error in volume of a rectangular box with dimensions $25 \times 30 \times 50$ if measurements are all $\frac{1}{10}$ longer or all $\frac{1}{10}$ shorter, and compute exact errors for comparison.

Sol: $V(x,y,z) = xyz$.

① When all measurements are $\frac{1}{10}$ longer:

Exact Error: $\Delta V = V(25+\frac{1}{10}, 30+\frac{1}{10}, 50+\frac{1}{10}) - V(25,30,50)$

$$= (25+\tfrac{1}{10})(30+\tfrac{1}{10})(50+\tfrac{1}{10}) - 25 \cdot 30 \cdot 50$$

$$= 25.1 \cdot 30.1 \cdot 50.1 - 25 \cdot 30 \cdot 50$$

$$= 37851.051 - 37500 = 351.051$$

Approximate Error: $\Delta V \approx dV = \nabla V \cdot d\vec{x}$

$$= \langle yz, xz, xy \rangle \cdot \langle dx, dy, dz \rangle$$

$dx, dy, dz = \frac{1}{10}$

$$= yz\,dx + xz\,dy + xy\,dz$$

$x = 25,\ y = 30,\ z = 50$

$$= 30 \cdot 50 \cdot \tfrac{1}{10} + 25 \cdot 50 \cdot \tfrac{1}{10} + 25 \cdot 30 \cdot \tfrac{1}{10}$$

$$= 150 + 125 + 75 = 350$$

② All measurements $1/10$ shorter than stated.

Approximate Error: $dV = \nabla V \cdot d\vec{x} = \langle yz, xz, xy \rangle \cdot \langle dx, dy, dz \rangle$

$$= yz\,dx + xz\,dy + xy\,dz$$

$\therefore$ Approximate error is: $= 30 \cdot 50 \cdot (-\frac{1}{10}) + 25 \cdot 50 \cdot (-\frac{1}{10}) + 25 \cdot 30 \cdot (-\frac{1}{10})$

$$= -150 - 125 - 75 = -350.$$

Exact Error: $\Delta V = V(24.9, 29.9, 49.9) - V(25, 30, 50)$

$$= 24.9 \cdot 29.9 \cdot 49.9 - 25 \cdot 30 \cdot 50$$

$$= 37151.049 - 37500 = -348.951$$